The Picard stack of the curve

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Lecture 3

The relative curve

$$E \to \mathbb{F}_q((\pi)), \quad E \to [E:\mathbb{Q}_p],\ \mathbb{F}_q = \mathcal{O}_E/\pi$$

Last time: $F|\mathbb{F}_q$ perfectoid field $\rightsquigarrow X_F / \mathrm{Spa}(E)$

Now: $\underbrace{\mathrm{Perf}_{\mathbb{F}_q}}_{\mathbb{F}_q\text{-perfectoid spaces}} \ni S \rightsquigarrow X_S$ E-adic space

" X_S = family of curves $(X_{b(s)})_{s\in S}$ "

$*$ $E = \mathbb{F}_q((\pi))$

$$Y_S = \mathbb{D}^*_S = \{0<|\pi|<1\} \subset \mathbb{A}^1_S$$

$$\varphi = \mathrm{Frob}_S \circlearrowright Y_S, \qquad Y_S \to \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}(E)$$

$$X_S := Y_S / \varphi^{\mathbb{Z}}$$

$*$ $[E:\mathbb{Q}_p] < +\infty$

$S = \mathrm{Spa}(R, R^+)$ affinoid perfectoid

$$Y_S = \mathrm{Spa}\left(W_{O_E}(R^+), W_{O_E}(R^+)\right) \setminus \underbrace{V\left(\pi[\varpi]\right)}_{\text{stabilizes this closed subset}}$$

$\varpi \in R^{\circ\circ} \cap R^\times$ pseudo-uniformizing element

$X_S = Y_S / \varphi^{\mathbb{Z}}$ $\quad$ φ induced by

$$\sum_{n \geq 0} [x_n] \pi^n \longmapsto \sum_{n \geq 0} [x_n^q] \pi^n$$

on $W_{O_E}(R^+)$

Can glue this construction for any $S \in \mathrm{Perf}_{\mathbb{F}_q} \leadsto X_S$

Rem: There is no structural morphism $X_S \to S$

char 0 when $E/\mathbb{Q}_p$ $\quad$ char. p

Nevertheless one can construct a continuous map

$$\begin{array}{c} |X_S| \\ \downarrow \\ |S| \end{array}$$

open and closed i.e. "X_S/S is proper smooth"

Background on the pro-étale topology

Étale cohomology of diamonds

$\left(\mathrm{Spa}(R_i, R_i^+)\right)_{i\in I}$ filtered projective system of affinoid perfectoid spaces. Then $\varprojlim_{i\in I} \mathrm{Spa}(R_i, R_i^+)$ exists in $\underbrace{\mathrm{Perf}}_{\text{Perfectoid spaces}} = \mathrm{Spa}(R_\infty, R_\infty^+)$ where

$$\begin{cases} R_\infty^+ = \widehat{\varinjlim_i R_i^+} \\ R_\infty = R_\infty^+\left[\frac{1}{\varpi}\right] \end{cases}$$

ϖ-adic completion

image of any pseudo uniformizing element $\varpi_i \in R_i$ for some i

Def: * $X \to Y$ in Perf is pro-étale if locally on X and Y it is of the form $\varprojlim_{i \geq i_0} S_i \to S_{i_0}$ with étale transition morphisms.

* A pro-étale morphism $X \xrightarrow{f} Y$ is a covering if $\forall V \subset Y$ open qc $\exists U \subset X$ open qc s.t. $f(U) = V$.

$\rightsquigarrow$ gives rise to the pro-étale topology.

Ex: $x \in X$, $\varprojlim_{U \ni x} U = \underbrace{\mathrm{Spa}\left(k(x), k(x)^+\right)}_{\text{set of generalizations of } x} \xhookrightarrow{\text{pro-étale}} X$

$\rightarrow$ very different from schemes: $\varprojlim_{U \ni x} U = \mathrm{Spec}(\mathcal{O}_{X,x})$

Ex: Normal $\mathbb{Q}_p$-rigid analytic spaces $\xhookrightarrow[\text{faithfull}]{\text{fully}}$ Pro-étale sheaves on $\mathrm{Perf}_{\mathbb{Q}_p}$

Moreover $\forall X$ $\exists$ $\underbrace{\tilde{X}}_{\text{perfectoid}} \longrightarrow X$ pro-étale cover like before

Then $\underbrace{\tilde{X} \times_X \tilde{X}}_{\text{perfectoid}} \underset{\text{pro-étale}}{\rightrightarrows} \tilde{X}$ is a pro-étale eq. relation

fiber product in $\widetilde{\mathrm{Perf}}$ not in the category of adic spaces

$\Rightarrow$ $X = \tilde{X}/R$ as a pro-étale sheaf

$=$ $\boxed{\text{algebraic space for the pro-étale topology}}$

Notation: A locally profinite set. $\underline{A}(S) = \mathcal{C}^0(|S|, A)$ defines a pro-étale sheaf on Perf. $\underline{A}$ = constant sheaf w/ value A.

Ex: * $\mathrm{Spa}(\mathbb{Q}_p) = \underbrace{\mathrm{Spa}(\mathbb{Q}_p^{cyc})}_{\text{perfectoid.}} / \underline{\mathbb{Z}_p^\times}$

* $\mathrm{Spa}(\mathbb{Q}_p\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle)$

$$= \mathrm{Spa}(\mathbb{C}_p\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle) / \underline{\mathbb{Z}_p(1)^d \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$$

* $$\mathring{\mathbb{B}}^1(1,1) \xrightarrow{\log} \mathbb{A}^1_{\mathbb{Q}_p}$$

$\mathbb{Q}_p/\mathbb{Z}_p(1)$ – pro-étale torsor of pro-étale sheaves

* $\mathcal{H} \simeq \mathrm{Spf}(W(\overline{\mathbb{F}}_p)[[h_1, \dots, h_{n-1}]])$ Lubin-Tate space of deformations of a 1-dim. height n formal p-divisible group over $\overline{\mathbb{F}}_p$ (= Def(supersingular elliptic curve $/\overline{\mathbb{F}}_p$) if $n=2$.

$$X_b \to \mathcal{H}_\eta \simeq \mathring{\mathbb{B}}^{n-1}$$

$GL_n(\mathbb{Z}/p^b\mathbb{Z})$ - Galois cover

↑ generic fiber

moduli space of level b level structures on the universal deformation

$$\underline{(\mathbb{Z}/p^b\mathbb{Z})^n} \xrightarrow{\sim} H^{univ}[p^b]_\eta$$

$$\begin{tikzcd}
\varprojlim_{b} X_b \arrow[d] \arrow[d, bend right=60, "\underline{GL_n(\mathbb{Z}_p)}\ \text{pro-étale torsor}"'] \arrow[dd, bend left=60, "\underline{GL_n(\mathbb{Q}_p)^1}\ \text{pro-étale torsor}"] \\
\mathcal{X}_\eta \arrow[d, "\pi_{GH}"] \\
\mathbb{P}^{n-1}
\end{tikzcd}$$

fibers
||
Hecke orbits

Gross-Hopkins period morphism given by the Hodge filtration of the universal def.

$\boxed{\text{The Picard Stack}}$

$Perf_{\mathbb{F}_q}$ + pro-étale topology

⚠ : the final object of $\widetilde{Perf}_{\mathbb{F}_q}$, $\underbrace{Spa(\mathbb{F}_q)}_{\text{perfect adic space not perfectoid}}$, is not representable

pro-étale topos

$\boxed{\textbf{Def}: S \in Perf_{\mathbb{F}_q},\ Pic(S) = \text{groupoid of line bundles} / X_{S}}$

Fact (not easy): $\boxed{\text{Pic is a stack on } \mathrm{Perf}_{\mathbb{F}_q}}$.

$S \in \mathrm{Perf}_{\mathbb{F}_q}$ $\quad \mathcal{L}$ line bundle $/X_S$

$$|S| \longrightarrow \mathbb{Z}$$
$$s \longmapsto \deg\left(\mathcal{L}|_{X_{k(s), k(s)^+}}\right)$$

is locally constant (not easy)

$$\Rightarrow \boxed{\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d}$$

open/closed substack

As a consequence of a result of Kedlaya-Liu one can prove:

$$\boxed{\text{Th: } \mathrm{Pic}^d \xrightarrow{\sim} \left[\mathrm{Spa}(\mathbb{F}_q)/\underline{E}^\times\right] \qquad \mathcal{L} \longmapsto \mathrm{Isom}(\mathcal{O}(d), \mathcal{L})}$$

classifying stack of pro-étale $\underline{E}^\times$-torsors

$\longrightarrow \mathcal{L} \in \mathrm{Pic}^d(S)$, $\exists\, \tilde{S} \to S$ pro-étale cover s.t. $\mathcal{L}_{|X_{\tilde{S}}} \simeq \mathcal{O}(d)$

$+\ H^0(X_S, \mathcal{O}_{X_S}) = \underline{E}(S) \Rightarrow \underline{\mathrm{Aut}}(\mathcal{O}(d)) = \underline{E}^\times$.

Rem: Thus, the coarse moduli space of Pic^d is a point ~~no geometry~~.

Next Step: $d \geq 1$. Div^d sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ defined by

$$\mathrm{Div}^d(S) = \left\{ (\mathcal{L}, u) \;\middle|\; \mathcal{L} \in \mathrm{Pic}^d(S),\ u \in H^0(X_S, \mathcal{L}) \text{ s.t. } \underset{\forall s \in S,\ u_{|X_{k(s), k(s)^+}} \neq 0}{} \right\} / \sim$$

$\longrightarrow$ this is a diamond (algebraic space for pro-étale topology)

$\downarrow$

Hilbert diamond of degree d effective divisors on the curve

$\longrightarrow$ Formula $\quad \mathrm{Spa}(E)^{\diamond}/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \mathrm{Div}^1$

Scholze's diamond of untilts

$\rightsquigarrow$ link between $\underbrace{\text{the curve}}_{\text{not defined over } \mathbb{F}_q}$ and $\mathrm{Div}^1/\mathbb{F}_q$

$\longrightarrow$ $\overline{\mathbb{Q}}_\ell$-local systems $/\mathrm{Div}^1_{\overline{\mathbb{F}}_q} \simeq \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_E)$

$\longrightarrow$ Study $\Sigma^d : (\mathrm{Div}^1)^d \longrightarrow \mathrm{Div}^d$

$$((\mathcal{L}_1, u_1), \dots, (\mathcal{L}_d, u_d)) \longmapsto (\mathcal{L}_1 \otimes \dots \otimes \mathcal{L}_d, u_1 \otimes \dots \otimes u_d)$$

$\longrightarrow$ Study $AJ^d : \mathrm{Div}^d \longrightarrow \mathrm{Pic}^d$.